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ESTIMATE OF THE TIME OF OCCURRENCE OF DISCONTINUITIES IN THE SOLUTION OF A BOUNDARY VALUE PROBLEM FOR A SECOND ORDER QUASILINEAR HYPERBOLIC SYSTEM

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Iu.K. ENGEL'BREKHT
(Tallin)
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An estimate is obtained for the time of occurrence of a discontinuity in the solutions of a hyperbolic system of quasilinear differential equations with zero initial and continuous boundary conditions. Examples are presented for one-dimensional gasdynamics and geometrically nonlinear elasticity theory problems.

Problems of the analysis of nonlinear transient waves in gasdynamics and elasticity theory result in the integration of quasilinear hyperbolic systems of differential equations. Problems of the occurrence of discontinuities in the solutions of such systems under continuous initial conditions (the Cauchy problem), as well as under continuous boundary conditions (the boundary value problem) have been examined in [1 – 6]. The most general result has been obtained in [4] for the Cauchy problem, for which upper and lower bounds for the time of occurrence of discontinuities in the solution of a second order system have been established by using a congruence theorem.

An analogous result is obtained herein for the boundary value problem by using the Jeffrey method [4].

1. The following system with two independent variables t and x is considered:

$$\frac{\partial}{\partial t} U + A \frac{\partial}{\partial x} U = 0, \qquad U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{1.1}$$

$$u_j = u_j(x, t), \qquad a_{ij} = a_{ij}(u_1, u_2, x, t)$$

For $t \geqslant 0$ $(0 \leqslant x \leqslant \infty)$ the behavior of the solution of the system (1.1) is investigated under the initial conditions

$$u_j(x, 0) = b_j, \qquad b_j = \text{const} \qquad (j = 1, 2) \tag{1.2}$$

and under the boundary condition

$$u_1(0, t) = \varphi_1(t) \quad \text{or} \quad u_2(0, t) = \varphi_2(t) \tag{1.3}$$

where $\varphi_j(t)$ are continuous functions satisfying the condition of matching the initial conditions (1.2) and the condition (1.3): $\varphi_j(0) = b_j$. The equations of the characteristics for the system (1.1) are

$$dx \,/\, dt = \lambda^{(i)} \qquad (i = 1, 2)$$

where $\lambda^{(i)}$ are roots of the equation $| A - \lambda I | = 0$.

Let us introduce the Riemann invariants r and s

$$r = \int q_1 l_1^{(1)} du_1 + \int q_1 l_2^{(1)} du_2$$
$$s = \int q_2 l_1^{(2)} du_1 + \int q_2 l_2^{(2)} du_2 \tag{1.4}$$

where $l_j^{(i)}$ are the left eigenvectors of the matrix A and q_i are integrating factors. The Riemann invariants r and s satisfy the following conditions on the characteristics:
On the characteristics of the $\lambda^{(1)}$ direction

$$\frac{dr}{d\alpha} = \frac{\partial r}{\partial t} + \lambda^{(1)} \frac{\partial r}{\partial x} = 0 \tag{1.5}$$

On the characteristics of the $\lambda^{(2)}$ direction

$$\frac{ds}{d\beta} = \frac{\partial s}{\partial t} + \lambda^{(2)} \frac{\partial s}{\partial x} = 0 \tag{1.6}$$

where α and β are parameters along the characteristics.

Let us determine the time of occurrence of a discontinuity in the quantity $\partial r / \partial t$. Differentiating (1.5) with respect to α, we obtain

$$\left(\frac{\partial}{\partial t} + \lambda^{(1)} \frac{\partial}{\partial x}\right)^2 r = \frac{\partial^2 r}{\partial t^2} + \frac{\partial \lambda^{(1)}}{\partial r} \frac{\partial r}{\partial t} \frac{\partial r}{\partial x} + \frac{\partial \lambda^{(1)}}{\partial s} \frac{\partial s}{\partial t} \frac{\partial r}{\partial x} + \frac{\partial^2 r}{\partial x \partial t} \lambda^{(1)} = 0 \tag{1.7}$$

Since

$$\frac{\partial^2 r}{\partial t^2} + \lambda^{(1)} \frac{\partial^2 r}{\partial x \partial t} = \frac{d}{d\alpha}\left(\frac{\partial r}{\partial t}\right), \qquad \frac{\partial r}{\partial x} = -\frac{1}{\lambda^{(1)}} \frac{\partial r}{\partial t}$$
$$\frac{\partial s}{\partial t} = \frac{\lambda^{(2)}}{\lambda^{(2)} - \lambda^{(1)}} \frac{ds}{d\alpha}$$

then (1.7) can be represented as

$$\frac{d}{d\alpha}\left(\frac{\partial r}{\partial t}\right) - \frac{\partial \lambda^{(1)}}{\partial r} \frac{1}{\lambda^{(1)}} \left(\frac{\partial r}{\partial t}\right)^2 - \frac{\partial \lambda^{(1)}}{\partial s} \frac{ds}{d\alpha} \frac{\lambda^{(2)}}{\lambda^{(2)} - \lambda^{(1)}} \frac{1}{\lambda^{(1)}} \frac{\partial r}{\partial t} = 0 \tag{1.8}$$

By introducing the new variable v_1 by means of the formulas

$$v_1 = \frac{\partial r}{\partial t} \exp g_1, \qquad g_1 = \int \frac{1}{\lambda^{(1)}} \frac{\lambda^{(2)}}{\lambda^{(2)} - \lambda^{(1)}} \frac{\partial \lambda^{(1)}}{\partial s} ds \tag{1.9}$$

we obtain

$$\frac{dv_1}{d\alpha} = \exp(-g_1) \frac{1}{\lambda^{(1)}} \frac{\partial \lambda^{(1)}}{\partial r} v_1^2 \tag{1.10}$$

under the boundary condition

$$v_1 [\alpha(0, t)] = F[\varphi_j(t)] \tag{1.11}$$

where $F(t)$ is a function which should be constructed on the basis of (1.9) by using the functions φ_1 or φ_2 given in the form of the boundary condition (1.3).

Let us select the appropriate congruence equation with a constant coefficient in the form

$$dV_1/dt = NV_1^2 \qquad (N = \text{const}) \tag{1.12}$$

Direct integration of Eq. (1.12) yields the solution

$$V_1 = \frac{V_{01}(\tau)}{1 - tNV_{01}(\tau)} \tag{1.13}$$

Here the function $V_{01}(\tau) = V_1(0, \tau)$ is defined by a boundary condition of the type (1.11). It follows from (1.13) that in the case $N > 0$ the solution of the congruence equation (1.12) has an infinite discontinuity for

$$t = T_*, \quad T_*\zeta = \tau + 1 / NV_{01}(\tau) \tag{1.14}$$

Now, let us clarify the time of occurrence of the discontinuity $t = t_*$ for (1.10). Let us assume that the coefficients a_{ij} of (1.1) and the Riemann invariants r and s are such that the quantity $\partial r / \partial t$ is uniquely determinable in terms of the boundary condition (1.3). Using the parametrization $\alpha(t) \equiv t$ and the congruence theorem [4], the upper t_+ and lower t_- bounds of the time of occurrence of the discontinuity in the solution of (1.10) on the characteristic in the $\lambda^{(1)}$ direction can be determined by taking the origin at $t = \tau$, $x = 0$.
The inequality

$$t_- \leqslant t_* \leqslant t_+ \tag{1.15}$$

is satisfied, where

$$t_+ = \tau + \lambda^{(1)}\left[\min \Phi(\tau, r, s) \frac{\partial r(t)}{\partial t}\Big|_{t=\tau}\right]^{-1} \tag{1.16}$$

$$t_- = \tau + \lambda^{(1)}\left[\max \Phi(\tau, r, s) \frac{\partial r(t)}{\partial t}\Big|_{t=\tau}\right]^{-1} \tag{1.17}$$

$$\Phi(\tau, r, s) = \frac{\partial \lambda^{(1)}(\tau)}{\partial r} \exp\left[g_1(r_0, s_0) - g_1(r_0, s(\tau))\right]$$

Here r_0 and s_0 are values of the Riemann invariants r and s at $t = \tau$, $x = 0$.

In the case $a_{ij} = a_{ij}(u_1, u_2)$, the characteristics in the $\lambda^{(1)}$ direction remain lines because of the initial conditions (1.2), and the condition $s = s_0$, and therefore $t_+ = t_-$ is satisfied along them. In this case, the critical time of occurrence of the discontinuity for waves being propagated in the direction of increasing x is determined by the formula

$$t_* = \tau + \lambda^{(1)}\left[\frac{d\lambda^{(1)}(\tau)}{\partial r} \frac{\partial r(t)}{\partial t}\Big|_{t=\tau}\right]^{-1} \tag{1.18}$$

Let us note that by investigating the problem of wave propagation in the direction of decreasing x for $t \geqslant 0$ $(0 \geqslant x \geqslant -\infty)$, we have analogously

$$t_* = \tau + \lambda^{(2)}\left[\frac{\partial \lambda^{(2)}(\tau)}{\partial s} \frac{\partial s(t)}{\partial t}\Big|_{t=\tau}\right]^{-1} \tag{1.19}$$

The formulas to determine the time of occurrence of discontinuities in the solution of the boundary value problem for the system of quasilinear differential equations (1.1) permit the determination of two stages in the development of the solution under the continuous boundary conditions (1.3), and the establishment of a qualitative estimate on the possibility of the appearance of a discontinuous solution. If $t_* < \tau$, then the solution remains continuous in the domain $t \geqslant 0$. $0 \leqslant x \leqslant \infty$. If $t_* > \tau$, then in this domain the solution is continuous for $t < t_*$ and has a discontinuity for $t > t_*$

2. Let us consider then an example from gasdynamics. The one-dimensional isentropic problem of gas motion reduces to (1.1) in which

$$u_1 = \rho, \; u_2 = u, \; a_{11} = a_{22} = u, \; a_{12} = \rho, \; a_{21} = c^2\rho^{-1} \tag{2.1}$$

Here u is the stream velocity, ρ the gas density, c the speed of sound, where $c^2 = A_0\gamma\rho^{\gamma-1}$, γ is the adiabatic index of the gas, and $A_0 = \mathrm{const}$. In the initial conditions

$$b_1 = \rho_0, \qquad b_2 = 0 \tag{2.2}$$

and the boundary conditions

$$u_2(0, t) = u(0, t) = \varphi_2(t) \tag{2.3}$$

In this case

$$\lambda^{(1)}, \lambda^{(2)} = u \pm c, \qquad r, \ s = \pm \frac{u}{2} + \frac{c}{\gamma - 1} \tag{2.4}$$

It is easy to see that

$$\frac{\partial \lambda^{(1)}}{\partial r} = \frac{\gamma + 1}{2}, \qquad \frac{\partial r}{\partial t} = \frac{\partial u}{\partial t} \tag{2.5}$$

Substituting the relationships (2.3) – (2.5) into (1.18), we obtain for the critical time

$$t_* = \frac{2c_0}{\gamma + 1}\left[\frac{\partial \varphi_2(0)}{\partial t}\right]^{-1} \tag{2.6}$$

where conditions (2.2) on the characteristic $dx / dt = c_0$ have been taken into account. Various authors [1] have obtained (2.6) earlier.

3. Let us consider the equation of motion of an elastic half-space taking account of a geometric nonlinearity in the strain tensor. This problem can also be reduced to (1.1). Let x be the Lagrangean space coordinate, and t the time. In this case

$$u_1 = \partial u / \partial x, \ u_2 = \partial u / \partial t, \ a_{11} = a_{22} = 0, \ a_{12} = -1, \ a_{21} = -\psi(u_1) \tag{3.1}$$
$$\psi(u_1) = c_0^2 (1 + 3u_1 + {}^3/_2 u_1^2)$$

Here u is the displacement, $c_0 = \text{const}$ is the velocity of wave propagation in the linear problem. The initial conditions (1.2) have the form

$$b_1 = 0, \qquad b_2 = 0 \tag{3.2}$$

The boundary condition corresponds to the force effect on the free plane of the half-space

$$u_1(0, t) = f_1(t), \qquad f_1(0) = 0 \tag{3.3}$$

In this case

$$\lambda^{(1)}, \ \lambda^{(2)} = \pm [\psi(u_1)]^{1/2} \tag{3.4}$$

The Riemann invariants are calculated after having expanded the integrands in a Maclauren series, taking account of the first two members

$$r, \ s = c_0 (u_1 + {}^3/_4 u_1^2 + {}^1/_4 u_1^3) \mp u_2 \tag{3.5}$$

It is easy to see that

$$\frac{\partial \lambda^{(1)}}{\partial r} = \frac{3}{2} c_0 (1 + u_1) \left(1 + 3u_1 + \frac{3}{2} u_1^2\right)^{-1/2} \frac{\partial u_1}{\partial r} \tag{3.6}$$

From (3.5) we obtain $r + s = 2c_0 (u_1 + {}^3/_4 u_1^2 + {}^1/_4 u_1^3)$, from which follows

$$\frac{\partial u_1}{\partial r} = \frac{1}{2}\left[c_0 \left(1 + \frac{3}{2} u_1 + \frac{3}{4} u_1^2\right)\right]^{-1} \tag{3.7}$$

Using the initial conditions (3.2), we obtain from (3.5)

$$\frac{\partial r}{\partial t} = 2c_0 \left(1 + \frac{3}{2} u_1 + \frac{3}{4} u_1^2\right) \frac{\partial u_1}{dt} \tag{3.8}$$

Substituting (3.6) – (3.8) into (1.18), we obtain the following approximate formula for the estimate of the time of occurrence of the discontinuity

$$t_* \cong \tau + \frac{2}{3}\left[1 + 2u_1(\tau) - \frac{1}{2}\ u_1^2(\tau)\right]\left[\left[\frac{\partial u_1(t)}{\partial t}\right]_{t=\tau}\right]^{-1} \tag{3.9}$$

If the discontinuity occurs on the first characteristic $dx / dt = c_0$, then the Riemann invariants (3. 5) are calculated exactly, and taking account of (3. 3), formula (3. 9) simplifies to

$$t_* = \frac{2}{3}\left[\frac{\partial f_1(0)}{\partial t}\right]^{-1} \tag{3.10}$$

Let us note that in this case the time of the discontinuity t_* is independent of the velocity of wave propagation c_0.

Let us consider the particular cases of formulating the boundary conditions (3. 3). Let us introduce the function $g(t)$ which has the properties

$$g(0) = 0, \quad g(t) > 0, \ \frac{\partial g(t)}{\partial t} > 0, \qquad \frac{\partial^2 g(t)}{\partial t^2} < 0 \quad \text{for} \quad t > 0$$

1. Let the boundary condition (3. 3) have the form

$$f_1(t) = g(t)H(t) \tag{3.11}$$

where $H(t)$ is the Heaviside function. Then the time of occurrence of the discontinuity in the solution is according to (3. 10)

$$t_* = \frac{2}{3}\left[\frac{\partial g(0)}{\partial t}\right]^{-1}$$

Let us note that under the boundary condition (3. 11) a tensile wave occurs on whose front a shockwave appears for $t = t_*$.

2. Let the boundary condition (3. 3) have the form

$$f_1(t) = -g(t)H(t) \tag{3.12}$$

In this case we have $t_* < 0$ according to (3. 10). The boundary condition (3. 12) defines a compression wave on whose front the solution remains continuous.

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